Noise scaling of phase synchronization of chaos

Victor Andrade,[1] Ruslan L. Davidchack,[1] and Ying-Cheng Lai[2]
[1]*Department of Physics and Astronomy, University of Kansas, Lawrence, Kansas 66045*
[2]*Departments of Mathematics and Electrical Engineering, Center for Systems Science and Engineering Research, Arizona State University, Tempe, Arizona 85287-1804*

(Received 1 October 1999)

We investigate the effect of noise on phase synchronization of coupled chaotic oscillators. It is found that additive white noise can induce phase slips in integer multiples of 2π's in parameter regimes where phase synchronization is observed in the absence of noise. The average time duration of the temporal phase synchronization scales with the noise amplitude in a way that can be described as *superpersistent transient*. We give two independent heuristic derivations that yield the same numerically observed scaling law.

PACS number(s): 05.45.Xt

The phenomenon of synchronous chaos has attracted much attention since the work of Pecora and Carroll in 1990 [1]. Typically, when two chaotic oscillators are coupled together, synchronization between them can occur when the coupling strength is large enough. Recently, a more delicate type of synchronization phenomenon was discovered by Rosenblum, Pikovsky, and Kurths [2]. This is the phase synchronization of chaotic oscillators which occurs at smaller coupling strength than that required for complete synchronization. Briefly, if trajectories in each chaotic oscillator can be regarded as a rotation, then the phase angle of the rotation increases steadily with time: $\theta(t)=\omega t+\phi(t)$, where ω is the average rotation frequency and $\phi(t)$ is a term characterizing chaotic fluctuations. As such, the rate of increase of phase can be modeled as a drift ω plus a zero mean chaotic process. In the absence of coupling, the phase angles of the two oscillators $\theta_1(t)$ and $\theta_2(t)$ are uncorrelated. That is, if one measures the difference $\Delta\theta(t)\equiv|\theta_1(t)-\theta_2(t)|$, one finds that $\Delta\theta(t)$ increases steadily with time. However, when a small amount of coupling is present, $\Delta\theta(t)$ can be confined within 2π, while the amplitudes of the rotations are still completely uncorrelated. The bifurcation that leads to this phase synchronization was subsequently investigated [3–5]. The ability of chaotic systems to have phase synchronization has implications on digital communication with chaos using the natural chaotic symbolic dynamics [6]. In such a case, it is highly desirable to suppress phase diffusions between chaotic communication channels to ensure proper timing for decoding.

In this Brief Report, we address to what extent phase synchronization can be observed in laboratory experiments by investigating the effect of noise on phase synchronization. Our principal results are (1) additive white noise, a type of noise encountered commonly in experimental situations, can induce phase slips in units of 2π between the coupled oscillators, which would otherwise be synchronized in phase in the absence of noise, and (2) the average time duration between successive phase slips appears to obey a scaling law with the noise amplitude ϵ,

$$\tau\sim\exp(K\epsilon^{-\alpha}), \tag{1}$$

where $\alpha>0$ is the scaling exponent depending on system parameters such as the coupling strength, and $K>0$ is a constant. An implication of this is that in the presence of only a small noise, the average time duration to observe phase synchronization can be extremely long. Phase synchronization is robust in this sense. In what follows, we first report our numerical experiments with the system of two coupled Rössler oscillators. We then give two independent heuristic derivations for scaling law (1).

We consider the following system of two coupled Rössler oscillators, the one that was originally used in Ref. [2] to first report phase synchronization: $dx_{1,2}/dt=-\omega_{1,2}y_{1,2}-z_{1,2}+C(x_{2,1}-x_{1,2})$, $dy_{1,2}/dt=\omega_{1,2}x_{1,2}+0.15y_{1,2}$, and $dz_{1,2}/dt=0.2+(x_{1,2}-10.0)z_{1,2}$, where C is the coupling strength, and we choose $(\omega_1,\omega_2)=(1.015,0.985)$, so that the two oscillators are slightly different in order to mimic a typical experimental situation where the oscillators cannot be perfectly identical. The Rössler chaotic attractor [7] has the property that its (x,y) variables represent a chaotic rotation with well-defined phase angles [2]. To compute the phase angles associated with the two oscillators, we find it convenient to use the polar coordinates (r,θ) to replace the (x,y) coordinates. In the cylindrical coordinate (r,θ,z), the Rössler equations become

$$\begin{aligned}
\frac{dr_{1,2}}{dt}&=0.15r_{1,2}\sin^2\theta_{1,2}\\
&\quad+[C(r_{2,1}\cos\theta_{2,1}-r_{1,2}\cos\theta_{1,2})-z_{1,2}]\cos\theta_{1,2},\\
\frac{d\theta_{1,2}}{dt}&=\omega_{1,2}+0.15\sin\theta_{1,2}\cos\theta_{1,2}\\
&\quad-\frac{1}{r_{1,2}}[C(r_{2,1}\cos\theta_{2,1}-r_{1,2}\cos\theta_{1,2})-z_{1,2}]\sin\theta_{1,2},\\
\frac{dz_{1,2}}{dt}&=0.2+(r_{1,2}\cos\theta_{1,2}-10.0)z_{1,2}.
\end{aligned} \tag{2}$$

When there is no coupling, the phase angles $\theta_1(t)$ and $\theta_2(t)$ are uncorrelated and, hence, the phase difference $\Delta\theta(t)=|\theta_2(t)-\theta_1(t)|$ increases steadily with time. Phase synchronization occurs when C is increased through the critical value $C_p\approx0.029$, in which we have $\Delta\theta(t)\leqslant2\pi$. The lower

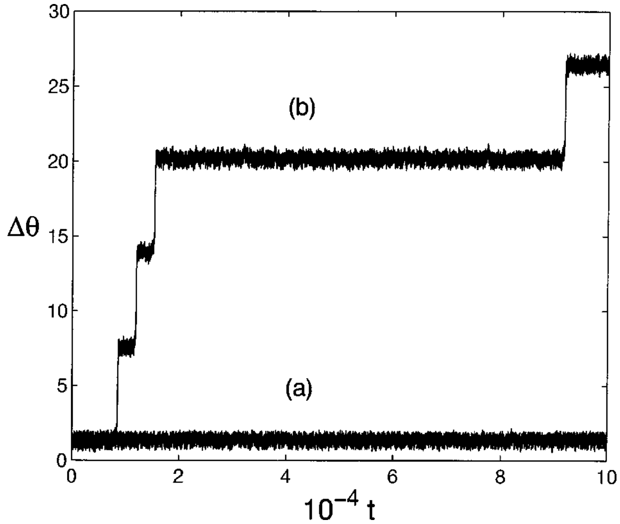

FIG. 1. For a system of two coupled Rössler oscillators: phase synchronization without noise (lower trace); and 2π phase slips induced by the noise of the amplitude $\epsilon \approx 10^{-3}$ (upper trace).

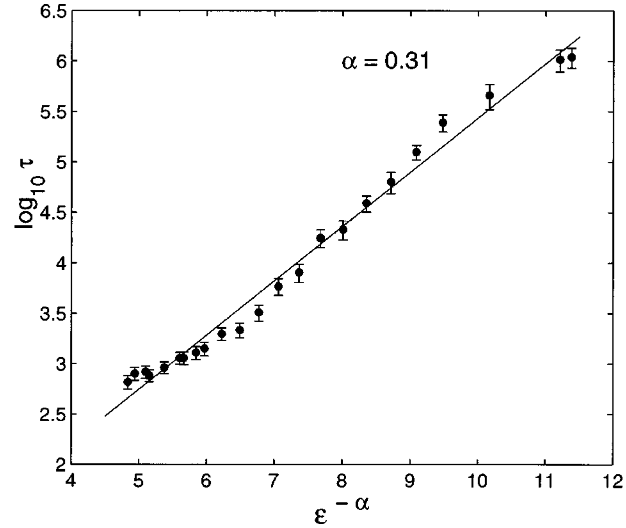

FIG. 2. For a system of two coupled Rössler oscillators at $C = 0.03$: $\log_{10}\tau$ vs $\epsilon^{-\alpha}$, where $\alpha \approx 0.31$ is a fitting parameter. Each point represents an average over 100 time intervals.

trace in Fig. 1 shows such a situation for $C = 0.03$, where $\Delta\theta(t)$ versus t is plotted. To model noise, we add different realizations of the terms $\epsilon\sigma_{r,\theta,z}(t)$ to each of the six variables in the coupled Rössler system [Eq. (2)] at each step of integration, where ϵ is the noise amplitude and the σ's are random variables uniformly distributed in $[-1, 1]$. The upper trace in Fig. 1 shows $\Delta\theta(t)$ versus t for $C = 0.03$, where the noise amplitude is $\epsilon \approx 10^{-3}$. We see that noise induces occasional phase slips in units of approximately 2π in $\Delta\theta(t)$. However, these phase slips are rare and become extremely infrequent as the noise amplitude is decreased further.

To quantify the 2π phase slips in Fig. 1, we compute how the average time interval τ [8] between successive phase slips changes as the noise amplitude is changed. For the parameter setting described above, we find that τ can be so prohibitively long that numerical computation of it becomes infeasible when the noise amplitude ϵ is smaller than, say, 10^{-4}. Figure 2 shows $\log_{10}\tau$ versus $\epsilon^{-\alpha}$ for $10^{-3.5} \lesssim \epsilon \lesssim 10^{-1.5}$ (approximately two orders of magnitude in ϵ), where $\alpha \approx 0.31$ is a fitting parameter. The approximate linear scaling behavior in Fig. 2 suggests scaling relation (1), which implies that the average time interval to observe the 2π phase slips behaves like e^{∞} as $\epsilon \to 0$. This is similar to the behavior of the superpersistent chaotic transients observed previously [9,10,3].

To qualitatively understand the scaling behavior in Fig. 2, we perform the following numerical experiment. First we set $\epsilon = 0$ and plot, in the coordinate $(r \equiv \sqrt{r_1^2 + r_2^2}, \Delta\theta)$, the attractors that result from two different initial conditions with $0 < \Delta\theta < 2\pi$ and $2\pi < \Delta\theta < 4\pi$, respectively, as shown in Fig. 3(a). Note that the variable $\Delta\theta$ is in fact a lifted angle variable [4], by which differences of the integer multiples of 2π are considered distinct. We see that initial conditions with 2π differences in $\Delta\theta$ result in attractors that live in different basins of attraction. Depending on the initial conditions, there is an infinite number of these attractors separated from each other by 2π in $\Delta\theta$. In the absence of noise, these attractors are completely isolated, corresponding to the situation of phase synchronization where $\Delta\theta$ remains within 2π if it starts with a value less than 2π. Next we examine the influence of noise on the phase-space structure in Fig. 3(a), as shown in Fig. 3(b) for $\epsilon = 10^{-2}$. We see that the basins of

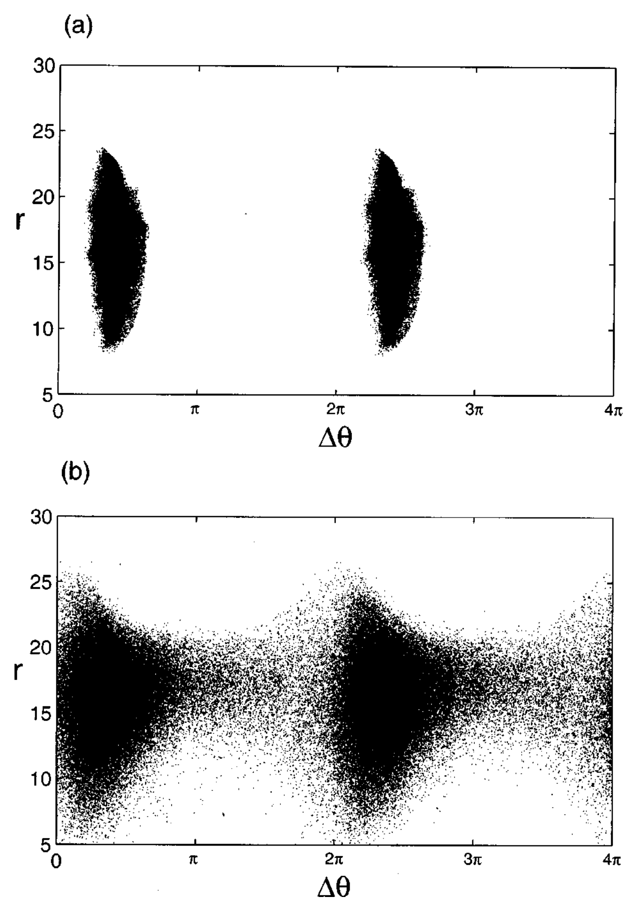

FIG. 3. Using the lifted phase variable $\Delta\theta$ for the system of two coupled Rössler oscillators: (a) two isolated phase-synchronized attractors in $0 < \Delta\theta < 2\pi$ and $2\pi < \Delta\theta < 4\pi$ in the absence of noise; and (b) tunneling between the previously isolated attractors due to noise.

attraction of the previously isolated attractors are now connected. There is now a nonzero probability that a trajectory can switch to different attractors separated by 2π in $\Delta\theta$, corresponding to the 2π phase slips observed in Fig. 1. The switch occurs when the trajectory falls into an open ''tunnel'' connecting the basins. The widths of these tunnels must be exponentially small, so that the probability for the trajectory to fall into a tunnel is extremely small, leading to the superpersisitent transient behavior observed in Fig. 2.

The numerically observed scaling law, as in Fig. 2, is only indicative of the dynamical characteristic of the noise-induced phase slips. It is difficult to extend the range of numerical computations because of the extremely long transient behavior between the phase slips. It is thus important to be able to derive heuristic theories to account for the scaling law. In what follows, we provide two independent theories, one based on the dynamical system approach and another on statistical mechanical methodology. Both theories yield the same scaling law.

(1) *Dynamical system approach*. Note that in Eq. (2), the scales of the time variation of the amplitude variables $r_{1,2}(t)$ and phase variables $\theta_{1,2}(t)$ are generally distinct. Since, on average, we have $\theta_{1,2}(t)\sim\omega_0 t$, we see that the phase angles $\theta_{1,2}(t)$ are ''fast'' variables. The amplitudes $r_{1,2}(t)$ are, however, slow variables because the Rössler chaotic trajectories have approximately a circularly rotational structure. Thus one can average over rotations of the phase angles to separate out the dynamics of the slow variables. Letting $\theta_{1,2}(t)=\omega_0 t+\phi_{1,2}(t)$, and performing averaging in the time interval $t\in[0,2\pi/\omega_0]$, yields [11]

$$\frac{d\Phi(t)}{dt}\approx 2\,\delta\omega+CG(r_1,r_2)\sin\Phi(t)+\text{white noise term}, \tag{3}$$

where $\Phi(t)\equiv\phi_2(t)-\phi_1(t)=\theta_2(t)-\theta_1(t)$, $\delta\omega\equiv\omega_1-\omega_2$, and $G(r_1,r_2)$ is a function that depends on the chaotic amplitudes $r_{1,2}(t)$. Equation (3) thus describes the dynamics of a chaotically driven limit-cycle oscillator. While the specific form of Eq. (3) is suitable for the system of coupled Rössler oscillators, we note a general feature of the phase-synchronization problem: *a limit-cycle oscillator driven by chaos*.

To facilitate analysis, we construct the following model of two-dimensional maps incorporating the general dynamical features of phase synchronization [11]:

$$\begin{aligned} x_{n+1}&=f(x_n),\\ \Phi_{n+1}&=\epsilon+pg_1(x_n)\Phi_n+g_2(x_n)\Phi_n^2+g_3(x_n)\Phi_n^3, \end{aligned} \tag{4}$$

where $f(x)$ is a chaotic map in which the variable x models the chaotic amplitudes in Eq. (3), $\epsilon\gtrsim 0$ models the combination of the small noise and the slight parameter mismatch between the two coupled chaotic oscillators, $g_{1,2,3}(x)$ are smooth functions, and p is a parameter that is proportional to the coupling strength. Assume that $f(x)$ generates a chaotic attractor with an infinite number of unstable periodic orbits embedded in it, and that phase synchronization occurs for $p>p_c$. In the Φ direction, these periodic orbits can be stable or unstable. For $p\gtrsim p_c$, all periodic orbits are stable in the Φ direction in the absence of noise, so Φ remains approximately constant (phase synchronization). Under the influence of noise, however, some of the periodic orbits become unstable in the Φ direction and, as such, a set of ''tongues'' opens at the locations of these periodic orbits, allowing the trajectory to escape from one approximately constant Φ state to another (2π phase slips). Typically, these orbits have low periods and the sizes of the tongues are exponentially small [9,10], which accounts for the extremely long time duration between the successive 2π phase slips. Let $\lambda>0$ be the Lyapunov exponent of the x chaotic attractor and let T be the time for a trajectory to tunnel through one of the tongues. We have, for the typical size of the opening of the tongue, $\delta\sim e^{-\lambda T}$. The average time between the successive phase slips is then

$$\tau\sim\frac{1}{\delta}\sim e^{\lambda T}. \tag{5}$$

The tunneling time T can be estimated by noting that when T is large, the map equation in Φ in Eq. (4) can be approximated as: $d\Phi/dt\approx\epsilon+[pg_1(x)-1]\Phi+g_2(x)\Phi^2+g_3(x)\Phi^3$, which yields

$$T\approx\int_0^{2\pi}\frac{d\Phi}{\epsilon+[pg_1-1]\Phi+g_2\Phi^2+g_3\Phi^3}. \tag{6}$$

The dependence of T on ϵ thus depends on the specific functions $g_{1,2,3}(x)$. For instance, since we know that most periodic orbits embedded in the x chaotic attractor are stable in the Φ direction, we have $pg_1(x)\lesssim 1$. A possible condition for a limit cycle oscillator is $g_2(x)\approx 1$ and $g_3(x)\approx 0$. Under these conditions, we have $T\sim\epsilon^{-1/2}$. If, however, we have $g_2(x)\approx 0$ and $g_3(x)\approx 1$, we have $T\sim\epsilon^{-2/3}$. In general, we expect $T\sim\epsilon^{-\alpha}$ and we obtain the scaling law (1).

(2). *Statistical mechanical approach* [12]. Note that Eq. (3), in the absence of noise, models the motion of a classical particle in a potential of the following form: $V(\Phi)=-2\,\delta\omega\Phi+CG(r_1,r_2)\cos\Phi$. When the coupling strength is large enough, the potential function $V(\Phi)$ possesses an infinite number of local minima separated by 2π in the phase variable Φ. On average, these minimum values of the potential function $V(\Phi)$ decrease linearly because of the linear term $-2\delta\omega\Phi$. The chaotic amplitude factor $G(r_1,r_2)$ models the fluctuations of the minimum potential values. When these minima are present, a particle starting near one of the local minima is trapped in its vicinity forever in a noiseless situation, signifying sustained phase synchronization. In the presence of noise, however, a particle originally in one of the local minima can be kicked into one of the adjacent minima, giving rise to a 2π phase jump. The probability for this to occur is $P\sim e^{-\Delta E/T}$, where ΔE is the typical height of the potential barrier that separates neighboring minima, and T is the ''temperature'' that is determined by the noise. Typically, we have $T\sim\epsilon^{\alpha}$, where $\alpha>0$. The average time for a 2π phase jump to occur is thus given by $\tau\sim 1/P\sim\exp(\Delta E\epsilon^{-\alpha})$, which is the scaling law [Eq. (1)].

In summary, we have studied the effect of small random noise on phase synchronization of coupled chaotic oscillators [13]. Under the influence of noise, indefinite phase synchronization is no longer possible [14]. Instead, 2π phase slips between the oscillators occur. When the noise amplitude is small, these phase slips are extremely rare. Thus we still expect to be able to observe phase synchronization for long times in well-controlled laboratory experiments where noise is small.

We thank Professor A. Pikovsky and S. Taherion for discussions. This work was supported by AFOSR under Grant No. F49620-98-1-0400 and by NSF under Grant No. PHY-9722156.

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990). For a recent review, see Chaos **7**, 520 (1997).

[2] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[3] A. S. Pikovsky, G. Osipov, M. G. Rosenblum, M. Zaks, and J. Kurths, Phys. Rev. Lett. **79**, 47 (1997).

[4] E. Rosa, E. Ott, and M. H. Hess, Phys. Rev. Lett. **80**, 1642 (1998).

[5] K. J. Lee, Y. Kwak, and T. K. Lim, Phys. Rev. Lett. **81**, 321 (1998).

[6] S. Hayes, C. Grebogi, and E. Ott, Phys. Rev. Lett. **70**, 3031 (1993); E. Rosa, S. Hayes, and C. Grebogi, *ibid.* **78**, 1247 (1997); E. Bollt and M. Dolnik, Phys. Rev. E **55**, 6404 (1997); E. Bollt, Y.-C. Lai, and C. Grebogi, Phys. Rev. Lett. **79**, 3787 (1997).

[7] O. E. Rössler, Phys. Lett. A **71**, 155 (1979).

[8] We find numerically (data not shown) that at large noise levels, say $\epsilon \gtrsim 10^{-2}$, the probability distributions of the time intervals of temporal phase synchronization are exponential. At smaller noise levels, the distributions cross over to become Lorentzian [5]. In both bases, an average time interval τ can be defined.

[9] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **50**, 935 (1983); Ergodic Theory Dyn. Syst. **5**, 341 (1985).

[10] Y.-C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).

[11] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).

[12] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).

[13] A dynamical quantity that characterizes the onset of phase synchronization is the Lyapunov spectrum [2]. Under the influence of noise, the spectrum as a function of the coupling parameter is typically shifted by an amount that is proportional to the noise amplitude, so that a larger coupling strength is required for phase synchronization to occur. However, there appears, to be no direct correspondence between the Lyapunov spectrum and the superpersistent transient scaling law.

[14] An alternative explanation for noise-induced 2π phase slips in chaotic phase synchronization is as follows. From our potential model, in the regime of phase synchronization ($C > C_p$), the depths of the potential wells are larger than the maximum amplitude of the chaotic fluctuations. Therefore, in the absence of noise, a particle trapped in one of the potential wells cannot move to the adjacent wells. Under the influence of noise, when the combined amplitude of the noise and chaotic fluctuations exceeds the depth of the potential well, 2π phase slips can occur. This is similar to the case before phase synchronization ($C < C_p$), where the depth of the potential well is smaller than the amplitude of the chaotic fluctuations. In this sense, intuitively, the phenomenon of noise induced 2π phase slips is similar to that observed before phase synchronization [5]. The parameter regime slightly before phase synchronization where the phase slips are rare is called the *nearly synchronous* regime [2]. The contribution of our paper, however, is a *quantitative* scaling analysis of the effect of noise on phase synchronization, which, to our knowledge, has not been reported before.